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Unstable Waves of Jet Flows With Density Inhomogeneity

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Instability of axisymmetric jet flows of a fluid having a radius-dependent density is investigated. The necessary condition for the existence of unstable waves depends not only on the velocity profile but also on the density gradient as well. Large density gradients, positive or negative, have stabilizing effects. The semi-circle theorem for amplified waves is valid in this case. It is shown by considering the top-hat type velocity profile that the velocity-dependent semi-circle bound is the best possible.

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UNSTABLE WAVES OF JET FLOWS WITH DENSITY INHOMOGENEITY

INTRODUCTION

The problem to be considered concerns linear unstable waves of axisymmetric jet flows in the presence of density inhomogeneities. Such flow phenomena occur when a jet is discharged into a stratified medium, e.g., pollutants and industrial waste discharged into the environment, cooling water discharged from power plants into rivers and lakes. and flow patterns generated by vehicles moving in the ocean. Investigating the instability characteristics of such jet flows is necessary to fully understand the overall behavior of the flow patterns and their corresponding effects on the environment. Changing the instability characteristics of the flow can mean controlling the flow patterns, for example, produced by vehicles moving in the ocean. As in the case of two-dimensional shear flows. axisymmetric jet flows may possess amplified waves due to the Kelvin-Helmholtz mechanism except that the formulation for jet flows is complicated by the absence of the Squire transformation and the consideration of cylindrical geometry. Keywords: JET FLOW, FLUID MECHANICS, WASTE WATER, EFFLUVIA, POWER PLANTS, WASTE DISPOSAL. (JES)

In a well-known paper concerning unbounded parallel flows of the jet-wake type. Batchelor & Gill (1962) presented a general mathematical analysis of the stability characteristics of axisymmetric flows of homogeneous fluids. A necessary condition for instability and a semi-circle theorem for possible unstable waves were derived. By discussing the governing equation near the jet axis, they were able to reach some general conclusions on the characteristic difference between the axisymmetric and nonaxisymmetric modes. Meanwhile Reynolds (1962) conducted an experimental study and observed axisymmetric condensations and puffs in a water tank. The axisymmetric mode of a jet column was also

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examined by Gill (1962) with a slightly viscous jet flow model and later by Mattingly & Chang (1974) with a coordinated theoretical and experimental investigation. The latter found that the dominant disturbance in the jet was an axisymmetric one. It is expected that initial disturbances will follow the direction of mean flows and the axisymmetric waves will in general first amplify for the onset of instability.

In this theoretical study, the effect of radial density variations on the stability of incompressible axisymmetric jet flows will be investigated. Both the necessary condition for instability and the semi-circle bound on amplified waves will be obtained through a unified integral representation. The results obtained here show that the necessary condition for the existence of amplified waves depends not only on the velocity profile, but also on the effect of density inhomogeneities. Both positive and negative density gradients have stabilizing effects. The semi-circle theorem for unstable waves can be extended to flows with radius-dependent density. For top-hat profiles of jet or wake type, the semi-circle bound is found to be the best possible.

GOVERNING EQUATIONS AND INSTABILITY CHARACTERISTICS

The jet flow to be considered consists of a radius-dependent axial velocity $W(r)$ in a fluid with density $\rho(r)$, and is confined within the annular region between two concentric cylinders located at $r = R_1$ and $r = R_2$. Neglecting the gravitational effects, one finds, within the framework of the normal mode analysis, that sinusoidal waves are governed by the two ordinary differential equations

$$(k^2 + \frac{m^2}{r^2})p = i\rho k[DW - (W - c)(Du + \frac{u}{r})], \tag{1}$$

$$Dp = -i\rho k(W - c)u \tag{2}$$

with $D = d/dr$.

Here u and p are the perturbations in radial velocity and pressure respectively. To consider only temporal instabilities, the azimuthal wave number m is an integer, the axial wave number k is real, while the phase velocity $c = c_r + ic_i$ is in general complex. Since the equations are invariant under complex conjugation, a non-zero c_i implies instability.

Combining equations (1) and (2) and eliminating the variable p, one finds a single differential equation governing instability as follows:

$$(W - c)\Big\{D[\rho q(Du + \frac{u}{r})] - \rho u\Big\} - D_*(\rho q DW)u = 0 \tag{3}$$

where $D_* = D - 1/r$ and $q = r^2/(m^2 + k^2r^2)$. Equation (3) and boundary conditions $u(R_1) = u(R_2) = 0$ represent an eigenvalue problem for the inviscid unstable waves of axisymmetric jet flows in a stratified medium. Reminiscent of the Rayleigh equation

encountered in the stability of two-dimensional shear flow, equation (3) possesses a first-order singularity for neutral disturbances and analytical solutions in terms of well-known functions are in general not possible except for some special flow profiles. We will emphasize the effect of density inhomogeneities and seek solutions to those flow profiles which admit exact solutions in terms of modified Bessel Functions. To do this, one must avoid the first order singularity in the equation as c_i vanishes.

Before obtaining exact solutions to equation (3) we will briefly re-examine the inviscid instability characteristics of the flow via an integral representation. The analysis in the present case is particularly simple and the principal steps of deriving these characteristics are reproduced here for the sake of completeness. To perform the analysis, we look for unstable waves with $c_i > 0$ and make the transformation $u = (W-c)^n \psi$. Here n is an arbitrary real constant. Multiplying the resultant equation by the complex conjugate $r\bar{\phi}$, integrating it over the flow domain, and applying the boundary conditions that ϕ vanishes at the solid boundaries of the inner and outer cylinders, one obtains

$$\int_{R_1}^{R_2} \Big\{ (W-c)^{2n}\rho[q|D^*\psi|^2 + |\psi|^2] - (W-c)^{2n-1}[(n-1)D(\rho Q)]r|\psi|^2$$

$$-(W-c)^{2n-2}[n(n-1)\rho q(DW)^2]|\psi|^2 \Big\} r dr = 0 \qquad (4)$$

where $Q = rDW/(m^2+k^2r^2)$ and $D^* = D + 1/r$. Letting $n = 0$ one finds, from the imaginary part of the resultant integral

$$c_i \int_{R_1}^{R_2} D(\rho Q) \frac{|r\psi|^2}{|W-c|^2} dr = 0 \qquad (5)$$

For instability, the expression $D(\rho Q)$ must vanish at least at one interior point within the flow domain, i.e.,

$$\frac{DQ}{Q} + \frac{D\rho}{\rho} = 0 \qquad (6)$$

The first term of the above equation is reminiscent of the inflexion point theory encountered in plane parallel flows, and has been discussed in detail in the study of homogeneous jet flows (Batchelor & Gill 1962). The second term depends solely on density inhomogeneities and will have significant contributions when the condition for instability of the otherwise homogeneous flow field is satisfied. It is shown from equation (6) and later via the exact solution for a special flow profile, that the necessary condition for amplified waves established for homogeneous fluids is not sufficient if density inhomogeneities are present. Cylindrical vortex sheets can be stable if large density gradients are present in the flow.

The refinement of condition (6) can be obtained by equation (5) and the real part of equation (4) for $n = 0$. One can easily show a strong necessary condition for instability is

$$D\left[\frac{\rho r DW}{m^2+k^2r^2}\right](W - W_s) \leq 0 \qquad (7)$$

where W_s is the mean velocity evaluated at a certain point inside the flow field where condition (6) is satisfied.

The semi-circle bound can easily be derived by combining the real and imaginary parts of integral (4) for $n = 1$. Imposing the upper and lower limits a and b for the mean axial velocity, one obtains

$$0 \geq \int_{R_1}^{R_2} (W - a)(W - b)\rho[q|D^*\psi|^2 + |\psi|^2]rdr$$

$$= \left\{[c_r - \frac{1}{2}(a + b)]^2 + c_i^2 - [\frac{1}{2}(a - b)]^2\right\} \int_{R_1}^{R_2} \rho[q|D^*\psi|^2 + |\psi|^2]rdr \tag{8a}$$

$$[c_r - \frac{1}{2}(a + b)]^2 + c_i^2 \leq [\frac{1}{2}(a - b)]^2 \tag{8b}$$

Thus the semi-circle theorem is valid in the presence of density inhomogeneities, saying that the complex wave velocity must lie inside a semi-circle with the diameter equal to the range of the velocity. The subsequent section will show that this semi-circle bound provides an exact solution for some particular flow profiles.

TWO EXACT SOLUTIONS AND THEIR SEMI-CIRCLE BOUNDS

To examine the validity of the earlier obtained instability characteristics and to gain some information on instability growth rates as a function of velocity, density and wave numbers, one must seek explicit solutions to equation (3) for some special flow configurations. We will consider those profiles for which solutions in terms of modified Bessel functions are possible. One of such profiles to investigate is

$$W = W_1, \quad \rho = \rho_1(\frac{r}{R})^{\sigma_1} \quad for \quad R_1 \leq r < R$$

$$W = W_2, \quad \rho = \rho_2(\frac{r}{R})^{\sigma_2} \quad for \quad R \leq r \leq R_2 \tag{9}$$

where $W_1, W_2, \rho_1, \rho_2, \sigma_1$ and σ_2 are constants. This flow represents a top-hat jet core with radius-dependent density surrounded by a fluid column with different density. Because the discontinuity in the velocity at the interface $r = R$ satisfies instability condition (6) for homogeneous fluids, this velocity profile will generally be unstable for all modes in uniform media via the Kelvin-Helmholtz mechanism. However, it will be demonstrated later that these unstable waves can be stabilized if large density gradients are considered.

The perturbations in radial velocity and pressure for the flow profile in equation (9) are given by

$$u_j = r^{-\frac{\sigma_j}{2}-1}\{A_j[\frac{\sigma_j}{2} + \frac{krI'_{\nu_j}(kr)}{I_{\nu_j}(kr)}]I_{\nu_j}(kr) + B_j[\frac{\sigma_j}{2} + \frac{krK'_{\nu_j}(kr)}{K_{\nu_j}(kr)}]K_{\nu_j}(kr)\}$$

$$p_j = -i\rho_j R^{-\sigma_j} k(W_j - c) r^{\frac{\sigma_j}{2}} \{A_j I_{\nu_j}(kr) + B_j K_{\nu_j}(kr)\} \tag{10}$$

with $\nu_j = \sqrt{m^2 + (\sigma_j/2)^2}$ for $j = 1, 2$. Here $I_{\nu_j}(kr)$ and $K_{\nu_j}(kr)$ are the modified Bessel functions of the first and second kind of order ν_j, and a prime denotes the total derivative with respect to the quantity shown.

By integrating equations (1) and (2) across the interface at $r = R$, both the kinematical and dynamical interfacial conditions are obtained as follows:

$$< \frac{u}{W - c} >= 0$$

$$< p >= 0 \tag{11}$$

where

$$< \phi >= \phi(R_{+0}) - (R_{-0}).$$

Making use of the boundary conditions at $r = R_1, R_2$ and applying the matching conditions (11) yield the secular relation for instability

$$\frac{(W_1 - c)[I_{\nu_1}(\kappa) - H_1 K_{\nu_1}(\kappa)]}{[\frac{\sigma_1}{2} + \frac{\kappa I'_{\nu_1}(\kappa)}{I_{\nu_1}(\kappa)}]I_{\nu_1}(\kappa) - H_1[\frac{\sigma_1}{2} + \frac{\kappa K'_{\nu_1}(\kappa)}{K_{\nu_1}(\kappa)}]K_{\nu_1}(\kappa)}$$

$$-\alpha \frac{(W_2 - c)[K_{\nu_2}(\kappa) - H_2 I_{\nu_2}(\kappa)]}{[\frac{\sigma_2}{2} + \frac{\kappa K'_{\nu_2}(\kappa)}{K_{\nu_2}(\kappa)}]K_{\nu_2}(\kappa) - H_2[\frac{\sigma_2}{2} + \frac{\kappa I'_{\nu_2}(\kappa)}{I_{\nu_2}(\kappa)}]I_{\nu_2}(\kappa)} = 0 \tag{12}$$

Here

$$H_1 = \frac{[\frac{\sigma_1}{2} + \frac{\kappa_1 I'_{\nu_1}(\kappa_1)}{I_{\nu_1}(\kappa_1)}]I_{\nu_1}(\kappa_1)}{[\frac{\sigma_1}{2} + \frac{\kappa_1 K'_{\nu_1}(\kappa_1)}{K_{\nu_1}(\kappa_1)}]K_{\nu_1}(\kappa_1)}$$

$$H_2 = \frac{[\frac{\sigma_2}{2} + \frac{\kappa_2 K'_{\nu_2}(\kappa_2)}{K_{\nu_2}(\kappa_2)}]K_{\nu_2}(\kappa_2)}{[\frac{\sigma_2}{2} + \frac{\kappa_2 I'_{\nu_2}(\kappa_2)}{I_{\nu_2}(\kappa_2)}]I_{\nu_2}(\kappa_2)}$$

where $\alpha = \rho_2/\rho_1, \kappa = kR, \kappa_1 = kR_1$, and $\kappa_2 = kR_2$. The expression H_1 vanishes as $R_1 \to 0$ and so does H_2 as $R_2 \to \infty$ to represent solutions for unbounded flows. Solving equation (12) for the wave velocity yields

$$c = \frac{E_1 W_1 - \alpha E_2 W_2 \pm (W_1 - W_2)(\alpha E_1 E_2)^{\frac{1}{2}}}{E_1 - \alpha E_2} \tag{13}$$

Here

$$E_1 = \frac{[I_{\nu_1}(\kappa) - H_1 K_{\nu_1}(\kappa)]}{[\frac{\sigma_1}{2} + \frac{\kappa I'_{\nu_1}(\kappa)}{I_{\nu_1}(\kappa)}]I_{\nu_1}(\kappa) - H_1[\frac{\sigma_1}{2} + \frac{\kappa K'_{\nu_1}(\kappa)}{K_{\nu_1}(\kappa)}]K_{\nu_1}(\kappa)}$$

$$E_2 = \frac{[K_{\nu_2}(\kappa) - H_2 I_{\nu_2}(\kappa)]}{[\frac{\sigma_2}{2} + \frac{\kappa K'_{\nu_2}(\kappa)}{K_{\nu_2}(\kappa)}]K_{\nu_2}(\kappa) - H_2[\frac{\sigma_2}{2} + \frac{\kappa I'_{\nu_2}(\kappa)}{I_{\nu_2}(\kappa)}]I_{\nu_2}(\kappa)}$$

It can be shown that $E_1 \geq 0$ and $E_2 \leq 0$ for all the values of $\kappa_1 \leq \kappa \leq \kappa_2$. Nonzero c_i is therefore expected for all modes except for very small or large density ratios. A liquid jet impinging into the atmosphere is of course stable for all modes even in the presence of the strong Kelvin-Helmholtz effects arising from the cylindrical vortex sheets. This stabilization phenomenon stems from the fact that instability condition (6) is violated by large density gradients even though DQ vanishes inside the flow field. One other interesting feature we like to point out is that all amplified waves marked by equation (13), in spite of the density variations, wave numbers and locations of the solid boundaries, lie exactly on the boundary described by equation (8b) with the stable solutions corresponding to $\alpha = 0$ and $\alpha = \infty$ at both ends of the semi-circle. This exactness may be explained by examining integral (8a) for the flow under consideration. The velocity profile for the two-region flow can be written as

$$W = W_1 + (W_2 - W_1)H(r - R) \qquad for \qquad R_1 \leq r \leq R_2 \tag{14}$$

where $H(r-R)$ is the Heaviside function. The equality of equation (8b) is obtained through the substitution of equation (14) for W in integral (8a). Note that the instability growth rate for fixed k first increases with increasing α, reaches a maximum at an intermediate value and then decreases with still larger α, with the maximum growth rate ($|W_2 - W_1|/2$) governed by

$$\frac{I_{\nu_1}(\kappa) - H_1 K_{\nu_1}(\kappa)}{[\frac{\sigma_1}{2} + \frac{\kappa I'_{\nu_1}(\kappa)}{I_{\nu_1}(\kappa)}]I_{\nu_1}(\kappa) - H_1[\frac{\sigma_1}{2} + \frac{\kappa K'_{\nu_1}(\kappa)}{K_{\nu_1}(\kappa)}]K_{\nu_1}(\kappa)}$$

$$+\alpha \frac{K_{\nu_2}(\kappa) - H_2 I_{\nu_2}(\kappa)}{[\frac{\sigma_2}{2} + \frac{\kappa K'_{\nu_2}(\kappa)}{K_{\nu_2}(\kappa)}]K_{\nu_2}(\kappa) - H_2[\frac{\sigma_2}{2} + \frac{\kappa I'_{\nu_2}(\kappa)}{I_{\nu_2}(\kappa)}]I_{\nu_2}(\kappa)} = 0 \tag{15}$$

The special case of $\sigma_1 = \sigma_2 = 0$ in equation (15) for unbounded flows is plotted in figure (1) which shows the characteristic difference between the axisymmetric mode and the asymmetric ones.

The second example whose instability characteristics we wish to study for axisymmetric perturbations is a three-region flow with the profile given as:

$$
\begin{aligned}
&W = W_1 && \rho = \rho_1 && 0 \le r \le R_1 \\
&W = \frac{R_2^{2-\sigma} - r^{2-\sigma}}{R_2^{2-\sigma} - R_1^{2-\sigma}} W_1 + \frac{r^{2-\sigma} - R_1^{2-\sigma}}{R_2^{2-\sigma} - R_1^{2-\sigma}} W_2 && \rho = \rho_2 (\frac{r}{R_2})^{\sigma} && R_1 \le r \le R_2 \\
&W = W_2 && \rho = \rho_3 && R_2 \le r < \infty
\end{aligned}
\tag{16}
$$

Here ρ_1, ρ_2, ρ_3 and σ are arbitrary constants. This profile can be used to model a jet exhausting into an environment of different density and velocity with the middle region representing the shear layer to allow transitions of the physical quantities of the jet to its surroundings. One can examine such flow instabilities by adjusting the parameters W_j, R_j, ρ_j and σ. The perturbation radial velocities and pressures for the flow in the inner and outer regions can be reduced from equation (10) by setting $m = 0$ and $\sigma_j = 0$, while the solutions in the middle region are given by

$$
u_2 = r^{\mu-1}[A I_\mu(kr) + B K_\mu(kr)]
$$

$$
\begin{aligned}
p_2 = i\rho_2 (\frac{r}{R_2})^{2(1-\mu)} r^{\mu-2} \frac{1}{k} \Big\{ & \frac{2\mu(W_2 - W_1)}{R_2^{2\mu} - R_1^{2\mu}} r^{2\mu} [A I_\mu(kr) + B K_\mu(kr)] \\
- [\frac{R_2^{2\mu} - r^{2\mu}}{R_2^{2\mu} - R_1^{2\mu}} W_1 + \frac{r^{2\mu} - R_1^{2\mu}}{R_2^{2\mu} - R_1^{2\mu}} W_2 - c] & \Big\{ A[\mu + \frac{kr I'_\mu(kr)}{I_\mu(kr)}] I_\mu(kr) + B[\mu + \frac{kr K'_\mu(kr)}{K_\mu(kr)}] K_\mu(kr) \Big\} \Big\}
\end{aligned}
\tag{17}
$$

where $\mu = 1 - \sigma/2$. Applying the matching conditions (11) at both interfaces $r = R_1$ and $r = R_2$ gives the secular relation

$$
\begin{vmatrix}
[F_1(W_1 - c) + \frac{2\mu\delta^{2\mu}(W_2 - W_1)}{1 - \delta^{2\mu}}] I_\mu(\kappa_1) & [G_1(W_1 - c) + \frac{2\mu\delta^{2\mu}(W_2 - W_1)}{1 - \delta^{2\mu}}] K_\mu(\kappa_1) \\
[F_2(W_2 - c) + \frac{2\mu(W_2 - W_1)}{1 - \delta^{2\mu}}] I_\mu(\kappa_2) & [G_2(W_2 - c) + \frac{2\mu(W_2 - W_1)}{1 - \delta^{2\mu}}] K_\mu(\kappa_2)
\end{vmatrix} = 0
\tag{18}
$$

where

$$
F_1 = -\mu + \frac{\kappa_1 I_0(\kappa_1)}{\alpha_1 I'_0(\kappa_1)} - \frac{\kappa_1 I'_\mu(\kappa_1)}{I_\mu(\kappa_1)}
$$

$$
F_2 = -\mu + \frac{\kappa_2 K_0(\kappa_2)}{\alpha_2 K'_0(\kappa_2)} - \frac{\kappa_2 I'_\mu(\kappa_2)}{I_\mu(\kappa_2)}
$$

$$
G_1 = -\mu + \frac{\kappa_1 I_0(\kappa_1)}{\alpha_1 I'_0(\kappa_1)} - \frac{\kappa_1 K'_\mu(\kappa_1)}{K_\mu(\kappa_1)}
$$

$$
G_2 = -\mu + \frac{\kappa_2 K_0(\kappa_2)}{\alpha_2 K'_0(\kappa_2)} - \frac{\kappa_2 K'_\mu(\kappa_2)}{K_\mu(\kappa_2)}
$$

$$
\delta = \frac{R_1}{R_2}, \quad \alpha_1 = \frac{\rho_2}{\rho_1} (\frac{R_1}{R_2})^{\sigma}, \quad \alpha_2 = \frac{\rho_2}{\rho_3}.
$$

Rearranging determinant (18) by using the identity

$$I'_\mu(z)K_\mu(z) - I_\mu(z)K'_\mu(z) = \frac{1}{z},$$

one finds the solution for the wave velocity

$$c_r = \frac{W_1 + W_2}{2} + \frac{\mu(W_2 - W_1)}{1 - \delta^{2\mu}} \frac{(F_1 + G_2\delta^{2\mu})I_\mu(\kappa_1)K_\mu(\kappa_2) - (G_1 + F_2\delta^{2\mu})K_\mu(\kappa_1)I_\mu(\kappa_2)}{F_1G_2I_\mu(\kappa_1)K_\mu(\kappa_2) - G_1F_2K_\mu(\kappa_1)I_\mu(\kappa_2)} \tag{19a}$$

$$c_i = \frac{\mu(W_2 - W_1)}{1 - \delta^{2\mu}} \frac{\sqrt{\Delta}}{F_1G_2I_\mu(\kappa_1)K_\mu(\kappa_2) - G_1F_2K_\mu(\kappa_1)I_\mu(\kappa_2)} \tag{19b}$$

and

$$\Delta = 4\delta^{2\mu} - \Big[(\frac{1-\delta^{2\mu}}{2\mu} + \frac{1}{G_2} - \frac{\delta^{2\mu}}{F_1})F_1G_2I_\mu(\kappa_1)K_\mu(\kappa_2) - (\frac{1-\delta^{2\mu}}{2\mu} + \frac{1}{F_2} - \frac{\delta^{2\mu}}{G_1})G_1F_2K_\mu(\kappa_1)I_\mu(\kappa_2)\Big]^2 \tag{19c}$$

Thus amplified waves exist whenever $\Delta > 0$. equations(19) for the particular case $W_2 = 0, \rho_1 = \rho_2 = \rho_3$ and $\sigma = 0$ reduce to the result investigated earlier by Michalke & Schade (1963) in their study of shear flow instability in uniform fluids except that β^4 should be dropped from their equations (91) and (94). It should be pointed out that the complex wave speed in their limiting case as R_1 approaches R_2 appears to be incorrect. The solution for unbounded cylindrical vortex sheets should have been recovered if the proper limiting process had been taken. The flow in that limiting case was certainly unstable for all axial wave numbers with the real wave velocity approaching the velocity of the center of the jet as $k \to 0$. The limiting process as $R_1 \to R_2$ reduces equations(19) to

$$c_r = \frac{W_1 + W_2}{2} + \frac{(W_1 - W_2)}{2} \frac{[I_0(\kappa)K_1(\kappa) - \alpha K_0(\kappa)I_1(\kappa)]}{[I_0(\kappa)K_1(\kappa) + \alpha K_0(\kappa)I_1(\kappa)]} \tag{20a}$$

$$c_i = \pm(W_1 - W_2)\frac{\sqrt{\alpha I_0(\kappa)K_0(\kappa)I_1(\kappa)K_1(\kappa)}}{I_0(\kappa)K_1(\kappa) + \alpha K_0(\kappa)I_1(\kappa)} \tag{20b}$$

with α representing the density ratio between the outer and inner regions. equations (20) can of course be reduced from equation (13) for the special case of $\sigma_1 = \sigma_2 = 0$ and $m = 0$ for unbounded axisymmetric waves. It in fact can be shown that the general expression (19a) for c_r possesses a limiting value W_1 for long axial wave lengths. This acts as a support to Batchelor & Gill's finding that long waves with axisymmetry travel with the speed of the center of the jet. The corresponding instability discriminant reduced from (19c) takes the form

$$\Delta = -\frac{2}{\alpha_1\alpha_2\mu^2}[\alpha_1\mu(1 - \delta^{2\mu}) + (1 - \delta^{2\mu})^2]\kappa_2^2 log\kappa_2 > 0 \tag{21}$$

Thus long waves are always unstable except for $k = 0$. For sufficiently large wave numbers, the asymptotic form of the modified Bessel functions reduces equation (19c) to

$$\Delta = -[\frac{G_1 F_2 + 2\mu(G_1 - F_2 \delta^{2\mu})}{1 - \delta^{2\mu}}]^2 < 0 \tag{22}$$

Therefore amplified waves cannot exist in their short wavelength range except for the limiting case $R_1 = R_2$ in which the maximum growth rates for uniform fluids occurs in the limit $k \to \infty$. These instability behaviors are plotted out in Figs. (2) and (3) for the special case of $\alpha_1 = \alpha_2 = \sigma = 1$ and $W_2 = 0$, with $\epsilon = R_2 - R_1$ representing the shear layer thickness. Both the velocity and density in this case vary linearly across the shear layer. Figure (2) shows the instability growth rates in comparison with the eigenvalue bound in their complex velocity domain. The amplified rate for fixed shear layer thickness ϵ first increases with increasing k, reaches a maximum value at an intermediate value of wave number, and decreases to zero at the relatively large value of k given by equation (19c) when the discriminant vanishes. These instability growth rates approach the semi-circle bound as the shear layer thickness diminishes and the maximum growth rate $|W_2 - W_1|/2$ for cylindrical vortex sheets is reached when $\epsilon = 0$. Note that the complex wave velocity expands to the left portion inside the semi-circle as the density ratio between the jet core and the surroundings decreases. For sufficiently small or large density ratios, the flow is stable against all disturbances. The neutral stability boundary and the maximum growth rates are shown in Figure (3) as a function of the wave number and the shear layer thickness. The figure suggests that, for non-zero shear layer thickness, disturbances for which $\epsilon > 2/k$ are stable, and the maximum growth rate occurs when $k\epsilon$ is of the order of one-tenth. It is also shown through this particular flow configuration that amplified waves with axisymmetry may exist for some slowly varying profiles in the long axial wavelength range.

CONCLUDING REMARKS

The instability characteristics of axisymmetric jet flows in inhomogeneous fluids were discussed. The exact solutions to the governing stability equation confirm these characteristics; in particular, the semi-circle theorem provides the best possible bound on all amplified waves of the top-hat type velocity profiles. The discharge of a jet into a stratified medium may produce some organized flow patterns which do not exist in a homogeneous environment. An understanding of the instability characteristics is of practical interest, because changing those characteristics of the flow can mean controlling the flow patterns, for example, produced by vehicles moving in the ocean.

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NOMENCLATURE

$c = c_r + ic_i$	=	complex phase velocity
D	=	d/dr
I_ν	=	Modified Bessel Function of the first kind of order ν
k	=	axial wave number
K_ν	=	Modified Bessel Function of the second kind of order ν
m	=	azimuthal wave number
p	=	pressure perturbation
r	=	radius
R	=	position of the interface
R_1, R_2	=	positions of solid boundaries
u	=	velocity perturbation in the radial direction
W	=	axial velocity
κ	=	kR
κ_1, κ_2	=	kR_1, kR_2
ψ	=	transformation of u
ρ	=	density
$\sigma, \sigma_1, \sigma_2$	=	parameters for density variations

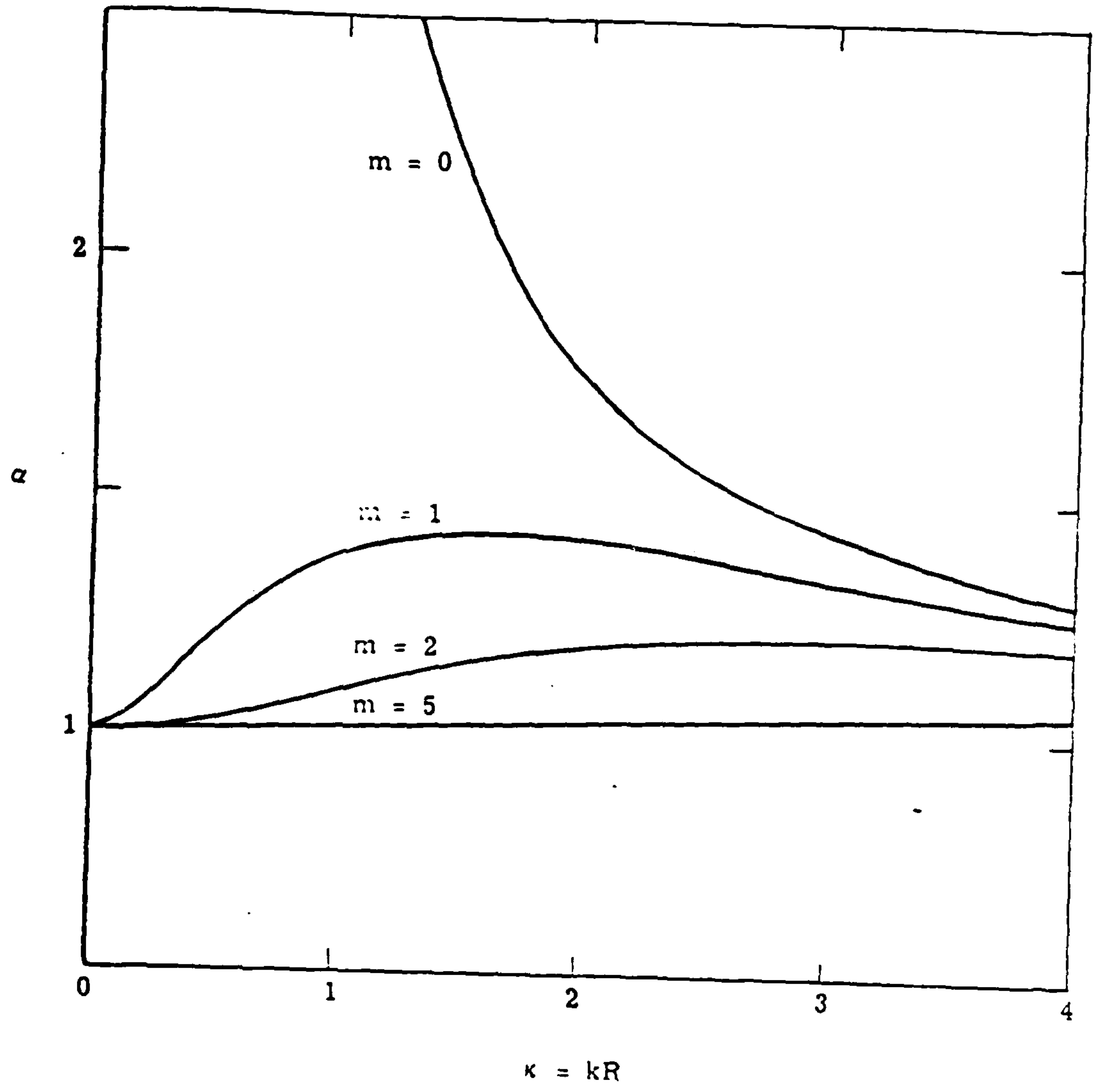

Figure 1. Curves of Maximum Growth Rates ($|W_2 - W_1|/2$) for $\sigma_1 = \sigma_2 = 0$.

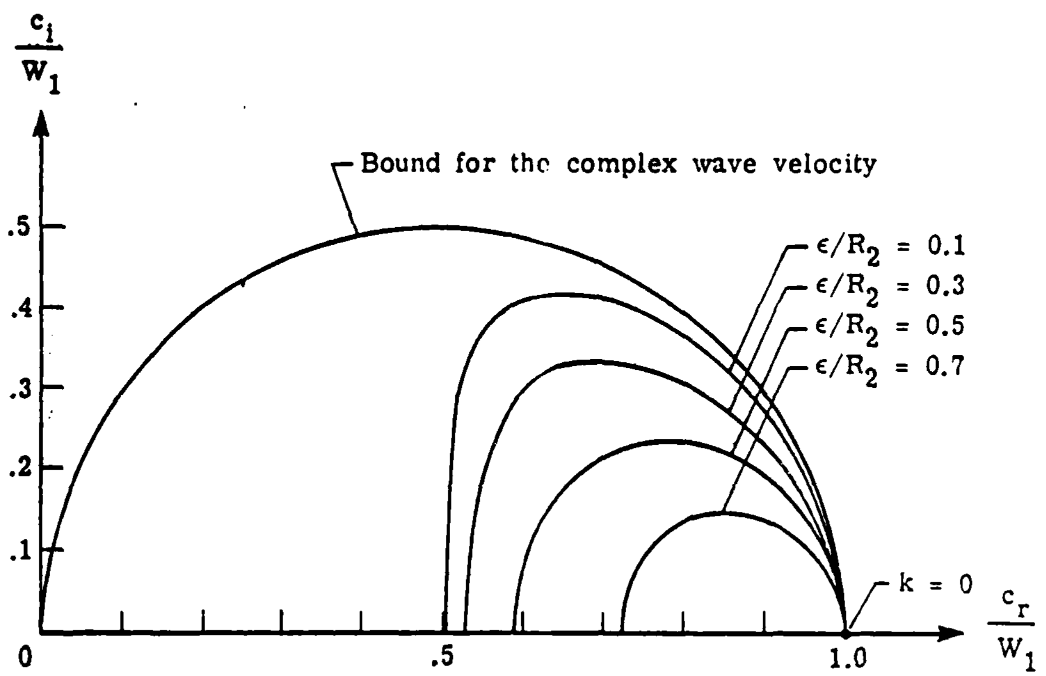

Figure 2. Instability Growth Rate versus the Shear Layer Thickness for the Special Case $\alpha_1 = \alpha_2 = \sigma = 1$ and $W_2 = 0$.

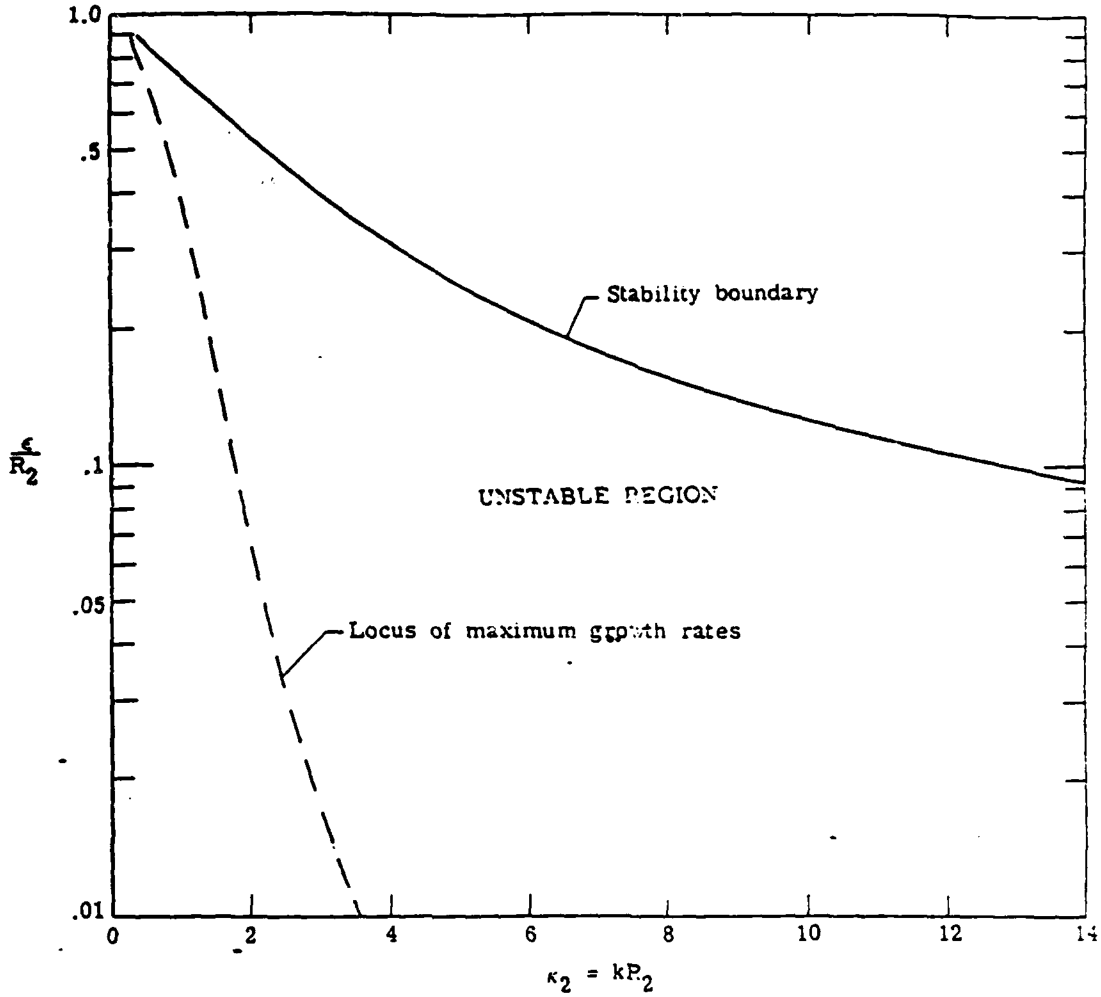

Figure 3. Variations of the Stability Boundary and the Maximum Growth Rates for the Special Case $\alpha_1 = \alpha_2 = \sigma = 1$ and $W_2 = 0$.